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Quantum mechanics as a geometric phase: phase-space interferometers

Alfredo Luis

Departamento de Óptica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain

E-mail: alluis@eucmax.sim.ucm.es

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Abstract
It is shown that basic quantum commutation relations are equivalent to a geometric phase. We propose two interferometric arrangements that are able to measure this quantum geometric phase via interference in phase space.

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1. Introduction

In very few years geometric phases have gained a pre-eminent status in physics [1]. Originally discovered in the framework of the quantum theory, the question of the possible quantum origin of some geometric phases has been well discussed [2]. However, the role they can play in the foundations of the quantum theory [3, 4] has received much less attention. It can be shown that basic commutation relations sustaining quantum mechanics are fully equivalent to a geometric phase arising after cyclic evolutions in phase space [4, 5]. Here we show that currently operative experimental arrangements can serve to detect and measure this quantum geometric phase via interference in phase space.

The equivalence between geometric phases and quantum physics might be regarded as the fundamentals of a novel formulation of the quantum theory. Given the relevance that geometric arguments have in natural science, this new formulation might eventually become comparable in importance to historic formulations. In particular, it might reveal novel correspondences between classical and quantum physics.

2. Quantum mechanics as a geometric phase

We focus on systems describable by an unbounded continuous degree of freedom represented by two canonically conjugate Cartesian variables q and p. The generalization to an arbitrary

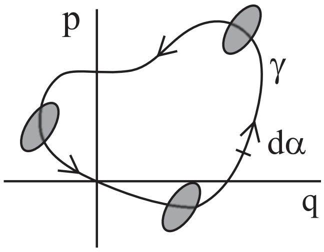

Figure 1. Closed path γ in phase space made of the composition of successive translations of value dα. The ellipses represent the quantum state of the system at three moments of its evolution along γ.

number of degrees of freedom is straightforward. In the quantum domain these variables become operators $\hat{q}$ and $\hat{p}$ satisfying the fundamental commutation relation

$$[\hat{q}, \hat{p}] = \mathrm{i}\hbar. \tag{2.1}$$

This lack of commutation of complementary variables is one of the cornerstones sustaining the quantum world and is directly responsible for basic quantum phenomena such as complementarity, uncertainty relations and so on. Suitable examples accessible to experiment in the quantum regime are the one-dimensional motion of a trapped ion (where q and p represent position and linear momentum, respectively) and a single mode of the electromagnetic field (where q and p are field quadratures).

A key ingredient in the context of geometric phases is cyclic evolution. In this work we will consider the evolution along closed loops γ in phase space such as the one represented in figure 1. In our case the phase space is a plane with q and p variables as rectangular coordinates. We can picture the evolution of the system via the representation of quantum states by quasi-probability distributions on phase space, such as the Wigner function [6]. Under the cyclic evolutions considered in this work it can be seen that the quasi-distribution evolves rigidly without experiencing any deformation or rotation. Therefore, all points describe the same trajectory γ at different locations.

In order to disclose the geometric phase we have to work in the standard formulation of quantum mechanics in Hilbert space. In such a case, phase-space translations are represented by the unitary displacement operators [6]

$$D(\alpha) = \mathrm{e}^{\mathrm{i}(p\hat{q} - q\hat{p})/\hbar} \tag{2.2}$$

where the complex quantitiy $\alpha = q + \mathrm{i}p$ parametrizes the displacement. We denote by $|\psi\rangle$ the (arbitrary) initial state of the system. During the evolution the system will be successively transformed into the displaced states $|\psi(\alpha)\rangle \propto D(\alpha)|\psi\rangle$ up to a phase factor. The first objective of this paper is to provide a simple explicit calculation of such a phase factor and its relation with the commutation relation (2.1).

We assume that the closed loop γ is an N-sided closed polygon with sides $\mathrm{d}\alpha_j$, $j = 1, 2, \ldots, N$, such that

$$\sum_{j=1}^{N} \mathrm{d}\alpha_j = 0. \tag{2.3}$$

An arbitrary curve can be suitably approached in the limit $N \to \infty$. The total transformation D_γ associated with γ is naturally given by

$$D_\gamma = D(\mathrm{d}\alpha_N) \cdots D(\mathrm{d}\alpha_1). \tag{2.4}$$

After any cyclic evolution the system returns to the initial state but acquires a phase in the state vector. We can compute such a phase noting that the commutation relation (2.1) implies the composition law [6]

$$D(\mathrm{d}\alpha)D(\alpha) = \mathrm{e}^{\mathrm{i}\mathrm{d}\beta} D(\alpha + \mathrm{d}\alpha) \tag{2.5}$$

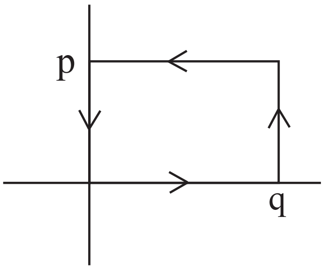

Figure 2. Rectangular closed path in phase space.

where

$$d\beta = \frac{1}{2\hbar}\,(q\,dp - p\,dq)\,. \tag{2.6}$$

This means that when computing the total transformation (2.4), each elementary displacement $d\alpha$ adds a differential phase $d\beta$. For a closed path this gives a total transformation proportional to the identity

$$D_\gamma = e^{i\beta} \tag{2.7}$$

where β is given by

$$\beta = \frac{1}{2\hbar}\oint_\gamma (q\,dp - p\,dq) = \frac{1}{\hbar}S \tag{2.8}$$

and S is the oriented area enclosed by γ.

The phase β depends only on the area of the circuit and is therefore a geometric phase. It does not depend on the form of the loop nor on the speed at which the transformation is executed. It is also worth pointing out that this phase appears irrespective of the state of the system and takes exactly the same value for each of them. In fact this is the geometric phase associated with the Heisenberg–Weyl group for one canonical pair of variables [4, 5].

This demonstrates that the very foundations of quantum physics imply the existence of a geometric phase. Next we show that the converse is also true: the quantum commutation relation (2.1) can be derived from this geometric phase. To show this we consider the simple rectangular circuit schematized in figure 2. We can naturally assume that every state is cyclic so that the corresponding transformation must be proportional to the identity, and, because of normalization of the state vectors, the constant of proportionality must have unit modulus

$$D_\gamma = e^{-ip\hat{q}/\hbar}e^{iq\hat{p}/\hbar}e^{ip\hat{q}/\hbar}e^{-iq\hat{p}/\hbar} = e^{i\beta} \tag{2.9}$$

where the phase β depends on the area of the circuit. From equation (2.9), the commutation relation (2.1) can be derived in the limits $q \to 0$, $p \to 0$ since in such a case both sides of the equality (2.9) can be approximated in the forms

$$D_\gamma \simeq 1 + \frac{qp}{\hbar^2}\,[\hat{q}, \hat{p}] \simeq 1 + i\beta \tag{2.10}$$

and this implies that

$$[\hat{q}, \hat{p}] = i\hbar^2 \lim_{q,p\to 0} \frac{\beta}{qp}. \tag{2.11}$$

Therefore, the quantum commutation relation (2.1) and the geometric phase (2.8) can be regarded as equivalent facts. In particular, $[\hat{q}, \hat{p}] \neq 0$ if and only if $\beta \neq 0$. It must be stressed that to derive β the Hilbert space structure underlying quantum mechanics is necessary from the start. Nevertheless, this does not unbalance the equivalence since the Hilbert space is also a prerequisite for imposing commutation relations between the linear operators $\hat{q}$ and $\hat{p}$.

Since this geometric phase is intimately connected with the foundations of the quantum theory, we can expect that it becomes unobservable when approaching the classical limit. For classical systems the typical areas S of phase-space loops are very large in comparison with $\hbar$. Since $\beta = S/\hbar$, in the classical limit small relative variations of S lead to large phase changes (short wavelength limit). Any real measurement unavoidably involves some kind of coarse-grained averages that will wash out the rapidly oscillating terms arising from the geometric phase. As expected, we conclude that β is not observable in the classical domain. We can recall that a very similar reasoning applies to the classical limit of the path-integral formulation of quantum mechanics.

We may consider the possibility of generalizing these results to other group structures, such as the $SU(2)$ group. In principle, it seems that it should always be possible to link the commutation relations satisfied by the infinitesimal generators to the corresponding geometric phase by using infinitesimal loops. For example, a direct relation between the $SU(2)$ geometric phase and the angular momentum commutation relations can be found in reference [7]. However, it is expected that such links will not be as simple as the one discussed in this work. This is because for the phase-space displacements, all the states are cyclic experiencing the same geometric phase and therefore the right-hand sides of equations (2.1), (2.7) and (2.9) are simply constants instead of operators. Furthermore, for the case analysed in this paper, the existence of a geometric phase is fully equivalent to the quantum nature of the system. We think that there is no natural extension of this result to arbitrary groups. For example, for the $SU(2)$ group the appearance of a geometric phase (the Pancharatnam phase) can be fully explained in the framework of classical physics, specifically in classical optical polarization [8, 9].

Finally, we can note that, in principle, this quantum phase β differs from the quantum phase concept arising when translating to the quantum domain classical phase variables such as $\varphi = \arg(q + \mathrm{i}p)$ [10]. Such a translation implies determination of how phases and phase shifts must be described and measured in the quantum domain. In this context β becomes a non-random phase shift that eventually should be disclosed by a suitable phase-detection scheme. The actual physical origin of β (classical or quantum, geometrical or dynamical) is not relevant [11]. We can also exclude a direct relationship between these two phase concepts in the case of the $SU(2)$ group, where the relevant phase variable φ is the phase difference (i.e. the azimuthal angle of the Poincaré or Bloch sphere). For example, it can be seen that different $SU(2)$ coherent states on the equator of the Poincaré or Bloch sphere (or even $SU(2)$ phase states [12]) can be in phase in the geometrical sense (they are connected by a single geodesic so that the solid angle they subtend vanishes) [9, 13], while they clearly represent different values of the phase difference [12]. In this context we can also mention the phase operator for two-state systems introduced in reference [14] that is suitably adapted to describe geometric phases.

3. Observation of the quantum geometric phase via interference in phase space

In this section we show that the quantum geometric phase β in equation (2.8) has observable consequences that can be directly verified experimentally. We demonstrate that this is possible in spite of the fact that all states experience the same geometric phase. First, we show in general terms how this observation can be achieved, and then we particularize to two possible practical implementations of the general scheme.

3.1. General approach

The natural framework where phases manifest themselves is interference. In our case we have to consider interferometric schemes in which the interfering paths are two different trajectories in phase space rather than in real space. To this end, the initial state of the system must be split into two components. The idea is to apply a different phase-space transformation to each component. One of them will be a cyclic transformation D_γ. The other component experiences no transformation (identity), so it acts as a reference for the phase acquired by the other one. The final state of the system after recombining the two paths will depend on β.

The splitting of the initial state requires the use of auxiliary degrees of freedom. The simplest choice is to describe these additional variables by using just two orthogonal states $|\pm\rangle$ which can represent spin components or the internal electronic state of an atom, etc. With the help of these variables we can consider the initial split state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|-\rangle + |+\rangle\right)|\psi\rangle \tag{3.1}$$

that experiences the unitary transformation

$$U = |-\rangle\langle-| + D_\gamma|+\rangle\langle+| \tag{3.2}$$

leading to the output state

$$U|\Psi\rangle = \frac{1}{\sqrt{2}}(|-\rangle + \mathrm{e}^{\mathrm{i}\beta}|+\rangle)|\psi\rangle. \tag{3.3}$$

The phase shift β is entirely contained in the auxiliary degrees of freedom. In order to detect β, we consider a measurement devised to determine the probability that the auxiliary degrees of freedom are in the state

$$|v\rangle = \frac{1}{\sqrt{2}}\left(|-\rangle + |+\rangle\right). \tag{3.4}$$

This occurs with probability

$$P = \frac{1}{2}\left(1 + \cos\beta\right). \tag{3.5}$$

This dependence of the final probability on β is the purely quantum interference phenomenon we were looking for. It is worth stressing that the probability P is completely independent of the system state $|\psi\rangle$. Moreover, the same result is obtained if the system is initially in an arbitrary mixed state.

In summary the auxiliary states $|\pm\rangle$ label two different paths in phase space. One of the paths experiences the transformation D_γ while the other acts as a reference. The final measurement recombines these two paths leading to the interference pattern in equation (3.5).

In what follows, we propose two experimental implementations of this interferometric scheme which are within the reach of present technology. In fact, both arrangements have been carried out successfully for different purposes.

3.2. Ion traps

First, we consider that the system variables q and p are the position and linear momentum, respectively, of the one-dimensional motion of a trapped ion. The auxiliary states $|\pm\rangle$ are two of the internal electronic levels of the ion. The selective displacement (3.2) can be carried out by applying pairs of off-resonant laser beams which drive two-photon-stimulated Raman transitions [15]. The final measurement (3.4) can be performed by detecting the internal state ($|+\rangle$ or $|-\rangle$) using field ionization detectors after applying a resonant laser pulse transforming

$|v\rangle$ into $|-\rangle$ ($\pi/2$ pulse). The probability of occupation of $|-\rangle$ can then be measured by driving the ion with a laser beam on resonance with a strong transition between $|-\rangle$ and a level different from $|+\rangle$. In case fluorescence is observed, the ion is in the state $|-\rangle$, otherwise the ion is in the state $|+\rangle$ [16].

All these steps have been actually carried out experimentally. According to reported data, it is possible to produce phase-space displacements with q and p up to 4×10^{-8} m and 4×10^{-26} kg m s^{-1}, respectively [15]. In a rectangular circuit, such as the one represented in figure 2, these values imply that β can be varied between -16 and 16, more than enough to suitably observe the quantum interference (3.5).

3.3. Cavity fields

Next, we examine a practical observation of this geometric phase in the area of quantum optics. As in the preceding example, we will closely follow the arrangements already carried out successfully [17]. In this case the system described by the q and p variables is an electromagnetic field mode in the microwave region contained in a high-Q cavity. A microwave source connected to the cavity can suitably displace the field state inside the cavity. The auxiliary degrees of freedom $|\pm\rangle$ are two internal electronic levels of a Rydberg atom that will cross the cavity. The atomic transitions and the cavity frequency are detuned enough so that there is no photon exchange. In such a case, the atom-field interaction produces a phase shift of the cavity field which depends on the internal atomic state. This action can be described by the unitary operator

$$V(\phi) = \mathrm{e}^{\mathrm{i}\phi a a^\dagger}|+\rangle\langle+| + \mathrm{e}^{-\mathrm{i}\phi a^\dagger a}|-\rangle\langle-| \tag{3.6}$$

where a is the complex amplitude operator for the field mode $a = (\hat{q} + \mathrm{i}\hat{p})/\sqrt{2}$, $\hat{q}$ and $\hat{p}$ being adimensional quadrature operators with $[\hat{q}, \hat{p}] = \mathrm{i}$, and ϕ is the one-photon phase shift which depends on the interaction time, the detuning and the Rabi frequency of the coupling between the atom and the cavity field.

The desired interference in phase space can be obtained as follows. Before the atom enters the cavity the initial field state $|\psi\rangle$ is displaced by $D(\alpha_1)$. The atom is prepared in a 50% coherent superposition of $|\pm\rangle$, as in equation (3.1), and then it crosses the cavity interacting with the field. At a given instant during the atom–field interaction, the field experiences another sudden displacement $D(\alpha_2)$. Afterwards the interaction continues. Once the atom leaves the cavity the initial state $|\Psi\rangle$ is transformed into $|\Psi'\rangle$,

$$|\Psi'\rangle = V(\phi_2)D(\alpha_2)V(\phi_1)D(\alpha_1)|\Psi\rangle \tag{3.7}$$

where ϕ_1 and ϕ_2 are, respectively, the one-photon phase shifts associated with the atom–field interaction before and after the second displacement $D(\alpha_2)$.

The output state can be expressed as

$$|\Psi'\rangle = W\frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}\phi a a^\dagger}|\psi\rangle|-\rangle + D_\gamma \mathrm{e}^{\mathrm{i}\phi a^\dagger a}|\psi\rangle|+\rangle\right) \tag{3.8}$$

where $W = \mathrm{e}^{\mathrm{i}\phi} D(\alpha_2 \mathrm{e}^{-\mathrm{i}\phi_2}) D(\alpha_1 \mathrm{e}^{-\mathrm{i}\phi})$,

$$D_\gamma = D\left(-\alpha_1 \mathrm{e}^{-\mathrm{i}\phi}\right) D\left(-\alpha_2 \mathrm{e}^{-\mathrm{i}\phi_2}\right) D\left(\alpha_2 \mathrm{e}^{\mathrm{i}\phi_2}\right) D\left(\alpha_1 \mathrm{e}^{\mathrm{i}\phi}\right) \tag{3.9}$$

and $\phi = \phi_1 + \phi_2$. The product of translations (3.9) is cyclic provided that

$$\alpha_1 \sin(\phi_1 + \phi_2) + \alpha_2 \sin\phi_2 = 0. \tag{3.10}$$

As in the preceding example, the geometric phase shift can be observed by measuring the population of the atomic levels $|\pm\rangle$ after applying a resonant $\pi/2$ pulse transforming $|v\rangle$ into

$|-\rangle$. We can appreciate that, in the general case,the field states associated with $|\pm\rangle$ in equation (3.8) are not the same. This would affect the visibility of the interference replacing equation (3.5) by

$$P = \frac{1}{2}\left[1 + \mathcal{V}\cos(\beta + \delta)\right] \tag{3.11}$$

where

$$\mathcal{V}\mathrm{e}^{\mathrm{i}\delta} = \mathrm{e}^{\mathrm{i}\phi}\langle\psi|\mathrm{e}^{\mathrm{i}2\phi a^\dagger a}|\psi\rangle. \tag{3.12}$$

Nevertheless, we can still have maximum visibility $\mathcal{V} = 1$ for any ϕ provided that the initial field state $|\psi\rangle$ is invariant under rotations in phase space. This is observed in the thermal states or the vacuum state.

For a proper choice of ϕ_1, ϕ_2, α_1 and α_2, the trajectory in phase space can be rectangular, such as the one illustrated in figure 2 (for example if $\alpha_2 = \pm\alpha_1$, $\phi_1 = -\pi/2$, $\phi_2 = \pi/4$). If the initial field state is a vacuum state it can be seen that the value of the geometric phase is of the order of the mean number of photons introduced in the cavity by the two displacements. Therefore a meaningful range of variation for β can be obtained even with a very low number of photons in the cavity field.

4. Conclusions

We have shown that there is a geometric phase fully equivalent to the quantum commutation relation between canonical operators. Such a quantum geometric phase can be experimentally observed with currently operating arrangements when used as interferometers in phase space.

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